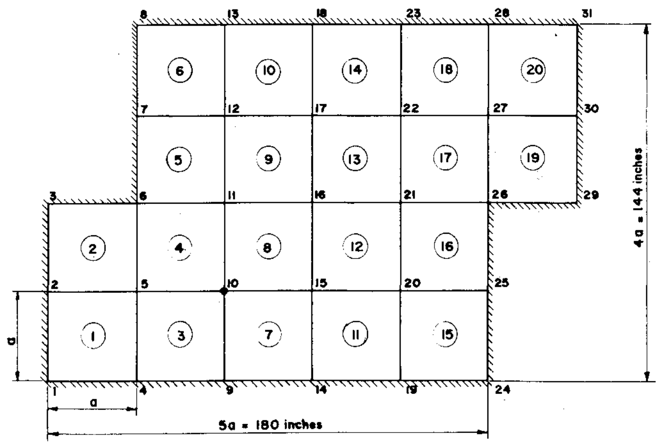


Fig. 3 Plate with fixed boundaries.

Table 2 Properties of compatibility matrices by DDR

| | Equilibrium matrix [B] | | Compatibility matrix [C] | |
|---|---|---|---|---|
| Parameter | Truss (81,101) | Plate (33,180) | Truss (81,101) | Plate (33,180) |
| Maximum bandwidth | 8 | 18 | 6 | 23 |
| Average bandwidth | 8 | 17.18 | 6 | 8.93 |
| Percentage sparsity | 4.82 | 5.85 | 5.94 | 2.04 |

2) The sparsity ratios of the compatibility matrix [C] are comparable to the equilibrium matrix for frame works. For plates, the sparsity ratio of compatibility matrix is smaller than the equilibrium matrix (2.04,5.85).

## Conclusions

1) In the IFM the compatibility conditions are generated from the deformation displacement relations of the structure without any reference to the popular redundants or basis determinate structure of the SFM.

2) The upper bound of the bandwidth of the compatibility condition depends on the element numbering of the discretization.

3) The bandwidth and sparsity ratio of the compatibility matrix [C] are comparable to those of the equilibrium matrix [B] for frame structures.

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# Thermal Effect on Frequencies of Coupled Vibrations of Pretwisted Rotating Beams

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## Introduction

In recent years, interest in the effect of temperature on solid bodies has increased greatly because of rapid developments in space technology, high-speed atmospheric flights, and nuclear energy applications. Sufficient work is available on coupled vibrations of beams.[1-5] It is well known[6] that, in the presence of a constant thermal gradient, the elastic coefficients of homogeneous materials become functions of the space variables. Fauconneau and Marangoni[7] have investigated the effect of the nonhomogeneity caused by a thermal gradient on the natural frequencies of simply supported plates of uniform thickness. Recently, Tomar and Jain[8] have studied the thermal effect on frequencies of a rotating wedge-shaped beam.

The purpose of this Note is to study the effect of a constant thermal gradient on coupled bending-bending-torsional vibrations of a pretwisted slender beam (that could represent a turbine blade of simple geometry) attached to a disk of radius $r_0$ as the disk rotates with an angular velocity $\Omega$. A method based on Rayleigh's quotient is used to obtain upper bounds of the frequencies corresponding to the first three vibration modes. The frequencies for various values of angle of pretwist, hub-radius change, and the temperature gradient are obtained for setting the angle $\pi/2$.

## Analysis and Equation of Motion

It is assumed that the beam is subjected to a steady one-dimensional temperature distribution along the length, i.e., in the $z$ direction.

$$T = T_0(1-\xi) \qquad (1)$$

where $T$ denotes the temperature excess above the reference temperature at any point at a distance $\xi = z/L$ and $T_0$ denotes the temperature excess above the reference temperature at the end $z = L$ or $\xi = 1$.

The temperature dependence of the modulus of elasticity for most engineering material is given by

$$E(T) = E_1(1-\gamma T) \qquad (2)$$

where $E_1$ is the value of the modulus of elasticity at the reference temperature, i.e., at $T = 0$ along the $z$ direction. Taking the temperature at the end of the beam, i.e., at $\xi = 1$, as the reference temperature, the modulus variation becomes

$$E(\xi) = E_1\{1-\alpha(1-\xi)\} \qquad (3)$$

where the temperature gradient $\alpha = \gamma T_0$ $(0 \le \alpha < 1)$.

The differential equations for coupled bending-bending-torsional vibrations of a pretwisted rotating beam are

$$\frac{\partial^2}{\partial z^2}\left(EI'_{yy}\frac{\partial^2 u}{\partial z^2} + EI'_{xy}\frac{\partial^2 v}{\partial z^2}\right) = -\rho A\frac{\partial^2}{\partial t^2}(u + \delta_y\theta) + \frac{\partial^2 M_x}{\partial z^2} \qquad (4a)$$

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$$\frac{\partial^2}{\partial z^2}\left(EI'_{xx}\frac{\partial^2 v}{\partial z^2}+EI'_{xy}\frac{\partial^2 u}{\partial z^2}\right)=-\rho A\frac{\partial^2}{\partial t^2}(v+\delta_x\theta)+\frac{\partial^2 M_y}{\partial z^2} \tag{4b}$$

$$\frac{\partial}{\partial z}\left[GJ\frac{\partial\theta}{\partial z}-\frac{\partial}{\partial z}\left(C_1\frac{\partial^2\theta}{\partial z^2}\right)\right]$$
$$=\rho A\delta_y\frac{\partial^2}{\partial t^2}(u+\delta_y\theta)+\rho A\delta_x\frac{\partial^2}{\partial t^2}(v+\delta_x\theta)I_\theta\frac{\partial^2\theta}{\partial t^2} \tag{4c}$$

where $u$ and $v$ are deflections of the shear center of the cross section at the point $z$ of the beam in $x$ and $y$ directions, respectively. $E$, $\rho$, $A$, $\theta$, $GJ$, $C_1$, and $I_\theta$ are the modulus of elasticity, density of the material, cross-sectional area of the beam, angle of twist, torsional rigidity, warping rigidity, and polar moment of inertia per unit length about the centroid, respectively. Here $I'_{xx}$, $I'_{yy}$, and $I'_{xy}$ are the product moment of cross-sectional area at point $z$ about the $x$, $y$, and $x$-$y$ axes, respectively. Also $\delta_x$ and $\delta_y$ are the distance between the shear center and centroid in the $x$ and $y$ directions, respectively, of a cross section at any point $z$. Further,

$$\frac{\partial^2 M_x}{\partial z^2}=\rho A\Omega^2\left[\left\{r_0(L-z)+\frac{1}{2}(L^2-z^2)\right\}\frac{\partial^2 u}{\partial z^2}-(r_0+z)\frac{\partial u}{\partial z}+\sin^2\chi u\right]$$

$$\frac{\partial^2 M_y}{\partial z^2}=\rho A\Omega^2\left[\left\{r_0(L-z)+\frac{1}{2}(L^2-z^2)\right\}\frac{\partial^2 v}{\partial z^2}-(r_0+z)\frac{\partial v}{\partial z}+\sin^2\chi v\right] \tag{5}$$

## Determination of Frequency Parameter

Equations (4) are now put in terms of the dimensionless variable $\xi=z/L$. Taking the solution of Eqs. (4) in the form

$$u(\xi,t)=Bf(\xi)e^{i\omega t},\quad v(\xi,t)=C\phi(\xi)e^{i\omega t},\quad \theta(\xi,t)=D\psi(\xi)e^{i\omega t} \tag{6}$$

where $B$, $C$, and $D$ are arbitrary constants and $f(\xi)$, $\phi(\xi)$, and $\psi(\xi)$ are functions of $\xi$ only, satisfying the following boundary conditions of the beam, i.e.,

$$f=\phi=\frac{\partial f}{\partial\xi}=\frac{\partial\phi}{\partial\xi}=\psi=\frac{\partial^2\psi}{\partial\xi^2}=0 \quad \text{at } \xi=0$$
$$\frac{\partial^2 f}{\partial\xi^2}=\frac{\partial^2\phi}{\partial\xi^2}=\frac{\partial^3 f}{\partial\xi^3}=\frac{\partial^3\phi}{\partial\xi^3}=\frac{\partial\psi}{\partial\xi}=\frac{\partial^3\psi}{\partial\xi^3}=0 \quad \text{at } \xi=1 \tag{7}$$

Substitution of Eqs. (3) and (6) into Eqs. (4) gives, after using equivalent expressions for $I'_{xx}$, $I'_{yy}$, and $I'_{xy}$, etc., from Ref. 9,

$$\frac{1}{AL^2}\left[\left\{RR_2\frac{\partial^4 P_i}{\partial\xi^4}+2(\alpha R_2-2\beta RK)\frac{\partial^3 P_i}{\partial\xi^3}-4\beta(\alpha K+\beta RR_1)\frac{\partial^2 P_i}{\partial\xi^2}\right\}r_i+\left\{RK\frac{\partial^4 q_i}{\partial\xi^4}+2(2\beta RR_1+\alpha K)\frac{\partial^3 q_i}{\partial\xi^3}+4\beta(\alpha R_1-\beta RK)\frac{\partial^2 q_i}{\partial\xi^2}\right\}s_i\right.$$
$$\left.-\frac{\rho\Omega^2L^2}{E_1}\left[\left\{\frac{r_0}{L}(1-\xi)+\frac{1}{2}(1-\xi^2)\right\}\frac{\partial^2 P_i}{\partial\xi^2}-\left(\frac{r_0}{L}+\xi\right)\frac{\partial P_i}{\partial\xi}+\sin^2\chi P_i\right]r_i=\lambda[P_ir_i+t_i\psi D] \quad i=1,2 \tag{8a}$$

$$\frac{J}{2(1+\nu)A}\left[R\frac{\partial^2\psi}{\partial\xi^2}+\alpha\frac{\partial\psi}{\partial\xi}\right]-\frac{C_0}{AL^2}\left[R\frac{\partial^4\psi}{\partial\xi^4}+2\alpha\frac{\partial^3\psi}{\partial\xi^3}\right]D$$
$$+\lambda\left[\delta_yfB+\delta_y^2\psi D+\delta_x\phi C+\delta_x^2\psi D+\frac{I_\theta}{\rho A}\psi D\right]=0 \tag{8b}$$

where

$$R=\{1-\alpha(1-\xi)\}$$

$$R_1=(-1)^i\left[\frac{I_{xx}-I_{yy}}{2}\cos2\beta\xi+I_{xy}\sin2\beta\xi\right]$$

$$R_2=\frac{I_{xx}+I_{yy}}{2}+R_1$$

$$K=(-1)^i\left[\frac{I_{xx}-I_{yy}}{2}\sin2\beta\xi-I_{xy}\cos2\beta\xi\right]$$

and

$$P_1=f,\quad q_1=\phi,\quad r_1=B,\quad s_1=C,\quad t_1=\delta_y$$
$$P_2=\phi,\quad q_2=f,\quad r_2=C,\quad s_2=B,\quad t_2=\delta_x$$

Here $\lambda=(\rho\omega^2L^2)/E_1$ is a frequency parameter.

For an approximate determination of $\lambda$, $f(\xi)$ and $\phi(\xi)$ are, respectively, chosen as the shape functions in two bending directions for the different modes of uncoupled bending vibrations, and $\psi(\xi)$ as the shape functions for different modes of uncoupled torsional vibrations of a uniform cantilever beam.

These shape functions satisfy boundary conditions (7) and are given as[10]

$$f(\xi)=\phi(\xi)=(\cosh\lambda_n\xi-\cos\lambda_n\xi)-\sigma_n(\sinh\lambda_n\xi-\sin\lambda_n\xi) \tag{9}$$

and

$$\psi(\xi)=\sin\left(\frac{2n+1}{2}\right)\pi\xi \qquad n=0,1,2,\ldots \tag{10}$$

where $\lambda_n$ and $\sigma_n$ are constants corresponding to the $n$th mode of vibration. Equations (8) can be solved for $\lambda$ but the result is a function of $\xi$, since $f$, $\theta$, and $\psi$ are not the exact shape functions. This difficulty can be overcome by multiplying Eqs. [(8a), $i=1$], [(8a), $i=2$], and (8b), respectively, by $f$, $\phi$, and $\psi$, and integrating from 0 to 1 with respect to $\xi$. The method results in the familiar Rayleigh quotient when applied to uncoupled problems, and is an extension of this method to the coupled problem. The following equations are obtained:

$$(a_1-\lambda a_{21}-a_{41})B-a_{11}C-\lambda a_{31}D=0$$
$$a_{21}B+(a_2-\lambda a_{22}-a_{42})C-\lambda a_{32}D=0$$
$$\lambda a_{31}B+\lambda a_{32}C+(a_{51}-a_{52}+\lambda a_{53})D=0 \tag{11}$$

where

$$a_i=\frac{1}{AL^2}\int_0^1\left\{RR_2\frac{\partial^4 P_i}{\partial\xi^4}+(2\alpha R_2-4\beta RK)\frac{\partial^3 P_i}{\partial\xi^3}-4\beta(\alpha K+\beta RR_1)\frac{\partial^2 P_i}{\partial\xi^2}\right\}P_i\mathrm{d}\xi$$

$$a_{1i}=\frac{1}{AL^2}\int_0^1\left\{RK\frac{\partial^4 q_i}{\partial\xi^4}+2(2\beta RR_1+\alpha K)\frac{\partial^3 q_i}{\partial\xi^3}-4\beta(\beta KR-\alpha R_1)\frac{\partial^2 q_i}{\partial\xi^2}\right\}P_i\mathrm{d}\xi$$

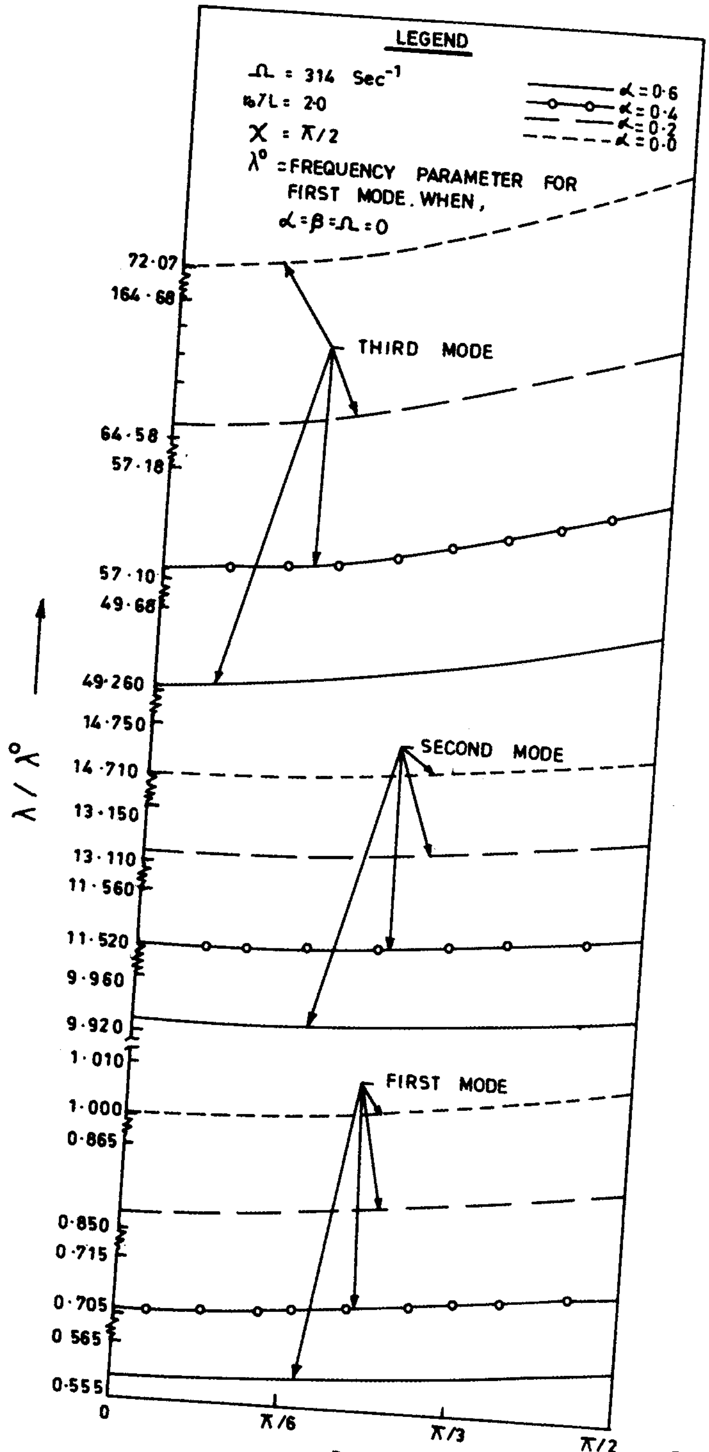


Fig. 1 Frequency parameter vs angle of pretwist for different values of temperature gradient.

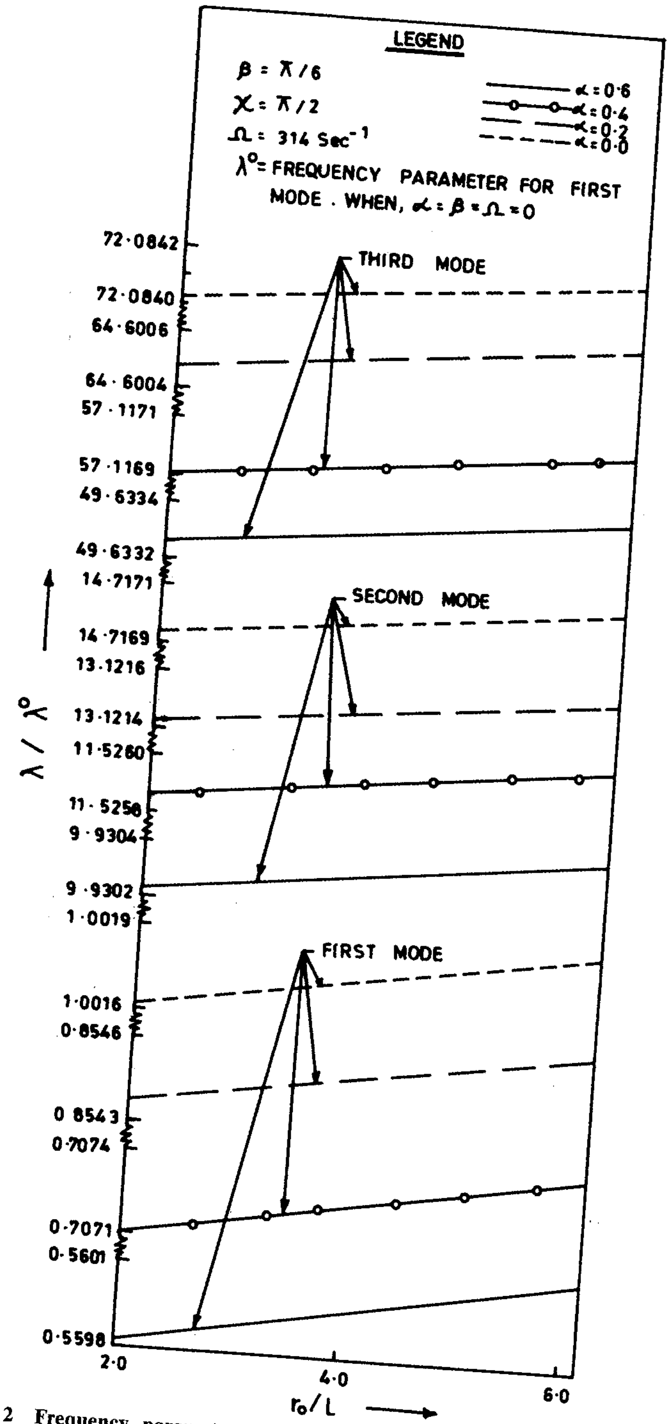


Fig. 2 Frequency parameter vs hub-radius change for different values of temperature gradient.

$$a_{2i}=\int_0^1 P_i^2\,\mathrm{d}\xi, \qquad a_{3i}=\int_0^1 t_i\psi P_i\,\mathrm{d}\xi$$

$$a_{4i}=\frac{\rho\Omega^2L^2}{E_I}\int_0^1\left[\left\{\frac{r_0}{L}(1-\xi)+\frac{1}{2}(1-\xi^2)\right\}\frac{\partial^2P_i}{\partial\xi^2}P_i - \left(\frac{r_0}{L}+\xi\right)\frac{\partial P_i}{\partial\xi}P_i+\sin^2\chi P_i^2\right]\mathrm{d}\xi$$

$$a_{51}=\frac{J}{2(1+\nu)A}\int_0^1\left[R\frac{\partial^2\psi}{\partial\xi^2}+\alpha\frac{\partial\psi}{\partial\xi}\right]\psi\,\mathrm{d}\xi$$

$$a_{52}=\frac{C_0}{AL^2}\int_0^1\left[R\frac{\partial^4\psi}{\partial\xi^4}+2\alpha\frac{\partial^3\psi}{\partial\xi^3}\right]\psi\,\mathrm{d}\xi$$

$$a_{53}=\int_0^1\left[(\delta_y^2+\delta_x^2)+\frac{I_\theta}{\rho A}\right]\psi^2\,\mathrm{d}\xi$$

For a nontrivial solution the determinant of the coefficients of Eqs. (11) must be zero:

$$\begin{vmatrix} a_1-\lambda a_{21}-a_{41} & a_{11} & -\lambda a_{31} \\ a_{12} & a_2-\lambda a_{22}-a_{42} & -\lambda a_{32} \\ \lambda a_{31} & \lambda a_{32} & a_{51}-a_{52}+\lambda a_{53} \end{vmatrix}=0 \quad (12)$$

**Table 1 Comparison of coupled bending-bending-torsion frequencies, Hz**

| Mode No. | Present results | Ref. 4, Table 2 | Ref. 2, Table A3 |
|---|---|---|---|
| 1 | 96.41 | 96.5 | 96.72 |
| 2 | 604.83 | 604.9 | 606.21 |
| 3 | 842.36 | 842.0 | 842.39 |
| 4 | 1095.74 | 1089.4 | 1076.48 |
| 5 | 1693.50 | 1694.9 | — |

This gives a third-degree polynomial in $\lambda$. The three values obtained would correspond to the upper bounds of frequencies in two bending directions and the third torsional motion of the particular mode.

## Results and Conclusions

The frequencies are computed from Eq. (12) and the cross section of the blade is taken as a semicircle of radius $a_0$ and thickness $t_l$. The physical constants are taken from Ref. 4 as follows:

$$L = 0.1524 \text{ m } (1.0 \text{ in.})$$

$$E_l = 2.0685 \times 10^{11} \text{ N/m}^2 \ (30 \times 10^6 \text{ psi})$$

$$t_l = 0.001727 \text{ m } (0.68 \text{ in.})$$

$$\rho = 7.8576 \times 10^3 \text{ kg/m}^2 \ (0.284 \text{ lb/in.}^3)$$

$$a_0 = 0.127 \text{ m } (0.5 \text{ in.})$$

$$\nu = 0.25$$

The smallest roots of Eq. (12), for various values of $\alpha$ corresponding to different values of $\beta$ and $r_0/L$, are plotted in Figs. 1 and 2, corresponding to the first three vibration modes. It is observed that the frequencies in all three vibration modes decrease with the increase in $\alpha$, and increase with increase in $\beta$ and $r_0/L$.

The results presented in Table 1 show good comparison with the results of Rao[11] using the Galerkin method when $\alpha = 0$.

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